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A RAYLEIGH-RITZ DETERMINATION OF UPPER AND LOWER BOUNDS
FOR THE DEFLECTION OF ORTHOTROPIC CANTILEVER BEAMS

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ABSTRACT

A numerical scheme based on polynomial approximation is

presented for the calculation of upper and lower bounds for the flexibility

coefficient of end-loaded orthotropic cantilever beams in a state of plane

stress, through use of the principles of minimum potential energy and

of minimum complementary energy. The degree-of-freedom of the two-

dimensional polynomials employed along with the elastic constants and

the depth-to-span ratio of the beam are external parameters of the

problem.

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INTRODUCTION

A formulation of the problem of the end-loaded cantilver beam,
which makes it possible to obtain upper and lower bounds for the flexibility
coefficient through the use of the principles of minimum potential energy
and of minimum complementary energy, has been reported earlier [1].
A later work [2] contains improved bounds obtained through the use of
arbitrary functions of the axial coordinate in the assumed admissible
expressions for displacements and stresses in the calculation of the energy
functionals. Although these improved bounds provide reliable corrections
to the elementary theory flexibility coefficient when the parameter μ
$\equiv G/\sqrt{EE_y}$ is not too large, the differences between the bounds becomes
significant for large values of μ.

In what follows we use a Rayleigh-Ritz procedure in conjunction with
the Lagrange multiplier method to improve the bounds obtained in [2].
We approximate the displacement components in the potential energy
calculations and the stress components in the complementary energy calcu-
lations by polynomials in the coordinate variables. The coefficients in
these polynomials are obtained by solving the simultaneous algebraic
equations which result upon extremizing the energy functionals subject to
the constraint conditions on these polynomials.

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Numerical results obtained employing sixth-degree polynomials in two variables show that for isotropic beams when $\nu = 1/3$, and the depth-to-length ratio is as high as 0.6 the maximum error in using the average value of the bounds instead of the actual value of the flexibility coefficient is less than 0.3%. Using polynomials of the same degree for orthotropic beams this error comes out to be less than 1% when $\nu = 1/2$ and $\mu = 100$ for all depth-to-span ratios.

FORMULATION

We briefly restate the problem of the end-loaded orthotropic beam as formulated in [1]. The inequality for the flexibility coefficient C reads

$$\frac{V^2}{2\tilde{I}_d} \leq C \leq \frac{V^2}{2\tilde{I}_s} \quad , \tag{1}$$

where V is the prescribed uniform end deflection and

$$\tilde{I}_d = \frac{1}{2} \int_{-c}^{c} \int_{o}^{a} \left\{ \frac{E\tilde{u}_{,x}^2 + 2\nu E_m \tilde{u}_{,x} \tilde{v}_{,y} + E_y \tilde{v}_{,y}^2}{1 - \nu^2} + G \left(\tilde{u}_{,y} + \tilde{v}_{,x} \right)^2 \right\} dx\,dy \quad , \tag{2}$$

$$\tilde{I}_s = V \int_{-c}^{c} \tilde{\tau}\,(0, y)\,dy - \frac{1}{2} \int_{-c}^{c} \int_{o}^{a} \left\{ \frac{\tilde{\sigma}_x^2}{E} - 2\nu \frac{\tilde{\sigma}_x \tilde{\sigma}_y}{E_m} + \frac{\tilde{\sigma}_y^2}{E_y} + \frac{\tilde{\tau}^2}{G} \right\} dx\,dy \quad , \tag{3}$$

with $2c$ and a representing, respectively, the depth and the length of the beam, and with $E_m = \sqrt{EE_y}$.

In equation (2), $\tilde{u}$ and $\tilde{v}$ are assumed axial and lateral displacement components which are differentiable functions satisfying the displacement boundary conditions

$$\tilde{u}\,(a, y) = \tilde{v}\,(a, y) = 0 \quad , \qquad \tilde{v}\,(0, y) = -V \quad , \tag{4}$$

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and in equation (3) $\tilde{\sigma}$ and $\tilde{\tau}$ are stress components which satisfy the equilibrium equations

$$\tilde{\sigma}_{x,x} + \tilde{\tau}_{,y} = 0 \quad , \qquad \tilde{\sigma}_{y,y} + \tilde{\tau}_{,x} = 0 \quad , \tag{5}$$

and the stress boundary conditions

$$\tilde{\tau}(x, \pm c) = \tilde{\sigma}_y(x, \pm c) = \tilde{\sigma}_x(0, y) = 0 \quad . \tag{6}$$

In what follows, we obtain upper and lower bounds for C by maximizing $\tilde{I}_s$ and by minimizing $\tilde{I}_d$ over a suitably restricted class of polynomials.

UPPER BOUND CALCULATION

We introduce the dimensionless quantities

$$\xi = \frac{x}{a} \quad , \qquad \eta = \frac{y}{c} \quad ; \qquad \rho = \frac{c}{a} \quad , \tag{7}$$

$$\left(\bar{\sigma}_x, \bar{\tau}, \bar{\sigma}_y \right) = \frac{a}{VE\rho} \left(\tilde{\sigma}_x, \frac{\tilde{\tau}}{\rho}, \frac{\tilde{\sigma}_y}{\rho^2} \right) \quad , \tag{8}$$

$$\lambda = \sqrt{\frac{E}{G}} \, \rho \quad , \qquad \mu = \frac{G}{E_m} \quad , \tag{9}$$

and have therewith, from (3), that $\tilde{I}_s = V^2 E \rho^3 \bar{I}_s$, where

$$\bar{I}_s = 2 \int_0^1 \bar{\tau}(0, \eta) \, d\eta - \int_0^1 \int_0^1 \left\{ \bar{\sigma}_x^2 - 2\nu\mu\lambda^2 \bar{\sigma}_x \bar{\sigma}_y + \mu^2 \lambda^4 \bar{\sigma}_y^2 + \lambda^2 \bar{\tau}^2 \right\} d\xi \, d\eta \quad . \tag{10}$$

We satisfy the equilibrium equations (5) identically by writing

$$\bar{\sigma}_x = F_{,\eta\eta} \quad , \qquad \bar{\tau} = -F_{,\xi\eta} \quad , \qquad \bar{\sigma}_y = F_{,\xi\xi} \quad , \tag{11}$$

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and assume F in the form

$$F = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \xi^m \eta^{2n-1} \quad , \tag{12}$$

where the A_{mn} are arbitrary constants, except for the relations

$$\sum_{n=1}^{N} (2n - 1) A_{mn} = 0 \quad , \qquad m = 1, 2, \ldots, M \quad , \tag{13}$$

$$\sum_{n=1}^{N} A_{mn} = 0 \qquad , \qquad m = 2, 3, \ldots, M \quad \cdot \tag{14}$$

which result upon satisfying the stress boundary conditions $\bar{\tau} = \bar{\sigma}_y = 0$ at $\eta = \pm 1$, with the condition $\tilde{\sigma}_x = 0$ for $\xi = 0$ being satisfied automatically.

Extremization of $\bar{I}_s$ with respect to the A_{mn}, subject to the constraint relations (13) and (14), is accomplished as follows.

We define a quantity $\bar{I}_s^*$ by

$$\bar{I}_s^* = \bar{I}_s + 2 \sum_{m=2}^{M} B_m \left(\sum_{n=1}^{N} A_{mn} \right) + 2 \sum_{m=1}^{N} C_m \left(\sum_{n=1}^{N} (2n - 1) A_{mn} \right) \quad , \tag{15}$$

with B_m and C_m being Lagrange multipliers.

Setting $\delta \bar{I}_s^* = 0$, we obtain

$$\sum_{k=1}^{M} \sum_{\ell=1}^{N} \left\{ \frac{\partial \bar{I}_s}{\partial A_{k\ell}} + 2B_k + 2C_k (2\ell - 1) \right\} \delta A_{k\ell} = 0 \quad , \tag{16}$$

where $B_1 = 0$, in view of the form of (15). From this we have further,

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$$\frac{1}{2} \frac{\partial \widetilde{I}_s}{\partial A_{k\ell}} + B_k + (2\ell - 1) C_k = 0 \quad , \qquad B_1 = 0 \quad , \qquad \begin{matrix} k = 1, \ldots, M \\ \\ \ell = 1, \ldots, N \end{matrix} \quad . \quad (17)$$

Equations (17), together with (13) and (14), represent a system of equations of just the right number to determine the unknown constants A_{mn}, B_m and C_m.

We next obtain an expression for $\overline{I}_{s\,max}$ in terms of the coefficients $A_{k\ell}$. To do this we make use of the fact that we may set $\delta A_{k\ell} = A_{k\ell}$ in equation (16) and write, using equations (13) and (14),

$$\sum_{k=1}^{M} \sum_{\ell=1}^{N} \frac{\partial \overline{I}_s}{\partial A_{k\ell}} A_{k\ell} = 0 \quad . \quad (18)$$

Considering the linear and quadratic terms present in equation (10), we may deduce from (10) in conjunction with (18) that

$$\overline{I}_{s\,max} = \int_o^1 \overline{\tau} (0, \eta) \, d\eta \quad . \quad (19)$$

With $\overline{\tau}$ from (11) and (12), this becomes

$$\overline{I}_{s\,max} = - \sum_{n=1}^{N} A_{1n} \quad . \quad (20)$$

We introduce the associated $\widetilde{I}_{s\,max}$ on the right of equation (1), and therewith obtain the upper bound formula

$$\frac{C_U}{C_o} = \frac{-1}{\sum_{n=1}^{N} A_{1n}} \quad , \quad (21)$$

where C_o is the elementary-theory flexibility coefficient $1/2\, E\rho^3$.

It remains to write the system (17) explicitly. To do this we introduce $\bar{\sigma}_x$, $\bar{\tau}$, and $\bar{\sigma}_y$ from equations (11) and (12) into equation (10) for $\bar{I}_s$ and carry out the differentiations with respect to $A_{k\ell}$ and the integrations with respect to ξ and η. In this way equation (17) becomes

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \frac{(2n-1)(2n-2)(2\ell-1)(2\ell-2)}{(m+k+1)(2n+2\ell-5)} \right.$$

$$- \nu\mu\lambda^2 \frac{(2n-1)(2n-2)k(k-1)+(2\ell-1)(2\ell-2)m(m-1)}{(m+k-1)(2n+2\ell-3)}$$

$$+ \mu^2\lambda^4 \frac{mk(m-1)(k-1)}{(m+k-3)(2n+2\ell-1)} + \lambda^2 \frac{mk(2n-1)(2\ell-1)}{(m+k-1)(2n+2\ell-3)} \left. \right\} A_{mn}$$

$$- B_k - (2\ell-1) C_k + \delta_{\ell k} = 0 \quad , \qquad \begin{array}{l} k = 1, 2, \ldots, M \\ \\ \ell = 1, 2, \ldots, N \end{array} \qquad (22)$$

again with $B_1 = 0$.

Setting $k = 1$, $\ell = 1$ in equation (22), we may sum the series which occurs in (21), as

$$\sum_{n=1}^{N} A_{1n} = \frac{C_1 - 1}{\lambda^2} \quad , \qquad (23)$$

and this enables us to write equation (21) in the final form

$$\frac{C_U}{C_o} = \frac{\lambda^2}{1 - C_1} \quad . \qquad (24)$$

It remains to solve (22), (13), and (14) in order to obtain the value of the Lagrange multipliers C_1.

We note that for $M = 1$, $N = 2$, the above procedure gives

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$C_U/C_o = 1 + 6\lambda^2/5$ which coincides, as it must, with a previously obtained upper bound [1].

LOWER BOUND CALCULATION

Introducing the nondimensional displacements

$$\bar{v} = \frac{\tilde{v}}{V} \quad , \qquad \bar{u} = \frac{\tilde{u}}{V\rho} \quad , \tag{25}$$

we write $\tilde{I}_d = V^2 E \rho^3 \bar{I}_d$ where

$$\bar{I}_d = \int_0^1 \int_0^1 \left\{ \frac{1}{1-\nu^2} \left(\bar{u}_{,\xi}^2 + \frac{2\nu}{\mu\lambda^2} \bar{u}_{,\xi} \bar{v}_{,\eta} + \frac{1}{\mu^2\lambda^4} \bar{v}_{,\eta}^2 \right) + \frac{1}{\lambda^2} \left(\bar{u}_{,\eta} + \bar{v}_{,\xi} \right)^2 \right\} d\xi \, d\eta \,. \tag{26}$$

Expressions for displcements are now assumed in the form

$$\bar{v} = \sum_{m=1}^M \sum_{n=1}^N a_{mn} (1-\xi)^m \eta^{2n-2} \quad , \qquad \bar{u} = \sum_{m=1}^{M'} \sum_{n=1}^{N'} b_{mn} (1-\xi)^m \eta^{2n-1} \,, \tag{27}$$

where, since $\bar{v}(0, \eta) = -1$, we have the constraint condition

$$\sum_{m=1}^M a_{mn} + \delta_{1n} = 0 \quad , \qquad n = 1, 2, \ldots, N \quad . \tag{28}$$

We obtain a_{mn} and b_{mn} by minimizing $\bar{I}_d$ with respect to these coefficients, subject to (28), that is by extremizing

$$\bar{I}_d^* = \bar{I}_d + 2 \sum_{n=1}^N c_n \left(\sum_{m=1}^M a_{mn} + \delta_{1n} \right) \,, \tag{29}$$

with the c_n being Lagrange multipliers.

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Setting $\delta \overline{I}_d^* = 0$ gives

$$\sum_{k=1}^{M} \sum_{\ell=1}^{N} \left\{ \frac{\partial \overline{I}_d}{\partial a_{k\ell}} + 2c_\ell \right\} \delta a_{k\ell} + \sum_{k=1}^{M'} \sum_{\ell=1}^{N'} \frac{\partial \overline{I}_d}{\partial b_{k\ell}} \delta b_{k\ell} = 0 \quad . \tag{30}$$

Equating the coefficient of each $\delta a_{k\ell}$ and $\delta b_{k\ell}$ in (30) equal to zero, we obtain the equations

$$\frac{1}{2} \frac{\partial \overline{I}_d}{\partial a_{k\ell}} + c_\ell = 0 \quad , \qquad \begin{array}{l} k = 1, 2, \ldots, M \\ \ell = 1, 2, \ldots, N \end{array} \quad , \tag{31}$$

$$\frac{1}{2} \frac{\partial \overline{I}_d}{\partial b_{k\ell}} = 0 \qquad , \qquad \begin{array}{l} k = 1, 2, \ldots, M' \\ \ell = 1, 2, \ldots, N' \end{array} \quad , \tag{32}$$

which, togeth er with (28), allow the determination of the unknown constants a_{mn}, b_{mn} and c_n.

We next obtain an expression for $\overline{I}_{d\,min}$ first in terms of the co-efficients a_{mn} and ultimately in terms of c_1 alone. To do this, we make use of the fact that we may set in equation (30) $\delta a_{k\ell} = a_{k\ell}$ and $\delta b_{k\ell} = b_{k\ell}$. and write

$$\sum_{k=1}^{M} \sum_{\ell=1}^{N} \frac{\partial \overline{I}_d}{\partial a_{k\ell}} a_{k\ell} + \sum_{k=1}^{M'} \sum_{\ell=1}^{N'} \frac{\partial \overline{I}_d}{\partial b_{k\ell}} b_{k\ell} + 2 \sum_{k=1}^{M} \sum_{\ell=1}^{N} c_\ell a_{k\ell} = 0 \quad . \tag{33}$$

Since $\overline{I}_d$ is a quadratic form, the first two terms in (33) add up to $2\overline{I}_{d\,min}$ and we have then

$$\overline{I}_{d\,min} = - \sum_{k=1}^{M} \sum_{\ell=1}^{N} c_\ell a_{k\ell} \quad . \tag{34}$$

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Equation (34) gives with the aid of the constraint equations (28),

$$\bar{I}_{d\,min} = c_1 \quad , \tag{35}$$

which, in connection with equation (1), gives the lower bound
formula

$$\frac{C_L}{C_o} = \frac{1}{c_1} \quad . \tag{36}$$

It remains to write equations (31) and (32) in explicit form. We
accomplish this by introducing $\bar{u}$ and $\bar{v}$ from (27) into the equation (26),
and by carrying out the differentiations with respect to $a_{k\ell}$ and $b_{k\ell}$
as well as the integrations with respect to ξ and η. This gives

$$\sum_{m=1}^{M'} \sum_{n=1}^{N'} \left\{ \frac{\lambda^2}{1-\nu^2} \frac{mk}{(m+k-1)(2n+2\ell-1)} + \frac{(2n-1)(2\ell-1)}{(m+k+1)(2n+2\ell-3)} \right\} b_{mn}$$

$$- \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \frac{\nu}{\mu(1-\nu^2)} k(2n-2) + m(2\ell-1) \right\} \frac{a_{mn}}{(m+k)(2n+2\ell-3)} = 0 \quad , \tag{37}$$

for $k = 1, 2, \ldots, M'$ and $\ell = 1, 2, \ldots, N'$, and

$$\sum_{m=1}^{M'} \sum_{n=1}^{N'} \left\{ \frac{\nu}{\mu(1-\nu^2)} m(2\ell-2) + k(2n-1) \right\} \frac{b_{mn}}{(m+k)(2n+2\ell-3)}$$

$$- \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \frac{1}{\mu^2 \lambda^2 (1-\nu^2)} \frac{(2n-2)(2\ell-2)}{(m+k+1)(2n+2\ell-5)} \right.$$

$$\left. + \frac{mk}{(m+k-1)(2n+2\ell-3)} \right\} a_{mn} = \lambda^2 c_\ell \quad , \tag{38}$$

for $k = 1, 2, \ldots, M$ and $\ell = 1, 2, \ldots, N$.

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Equation (37), (38) and (28) must now be solved in order to obtain the value of c_1 in (36). Numerical results reported in the next section are obtained by setting $M' = M - 1$, $N' = N$, so as to have an equal number of a-terms and b-terms in the experssions for the shear-strains which, numerical evidence indicates, is the most effective way to carry out this work. We note that the choice $M = 3$, $N = 1$ corresponds to the "semi-elementary" polynomial used in the calculation of the lower bound, $C_L / C_o = 1 - \nu^2 + \lambda^2$ [1].

DISCUSSION

In Table 1, values of upper and lower bounds are presented for isotropic beams when $\nu = 1/3$, and $c/a = 0.3$ and 0.5, as functions of the integers M and N representing the number of degrees-of-freedom of the associated admissible polynomials. The Hilbert-type matrices involved in the numerical work, require double precision arithmetic in the solution procedures for large values of M and N. The results show that in the extreme case of a beam which is as deep as it is long ($c/a = 0.5$), the maximum percentage error in employing the mean value of the present bounds with $M = N = 6$, instead of the _actual_ flexibility coefficient is only 0.22%.

Numerical values of the upper and lower bounds obtained for orthotropic beams, with the large value $\mu = G/E_m = 100$ when $\nu = 1/2$, are plotted against $\lambda = \sqrt{E/G}\ c/a$ in Figure 1. To obtain a

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maximum error of less than one percent it is found necessary to use

M = N = 6 in both upper and lower bound calculations. These results

are represented by the curves U66 and L66. For the sake of comparison,

we also include the bound curves U2 and L4 obtained in [2]. The

bounds U12 and L31 are equivalent to the earlier results obtained

in [1]. The bound curves U66 and L66 show that there is a decrease

in flexibility for small λ. As shown previously in [2], this decrease is

due to the end constraint v = 0 at x = a. The shear deformation effects

dominate when λ is large and the flexibility progressively increases with

increasing values of λ.

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         Int. J. Solids Structures 11:961-971, 1975.

Table 1. Convergence of the bounds for
isotropic beams when $\nu = 1/3$.

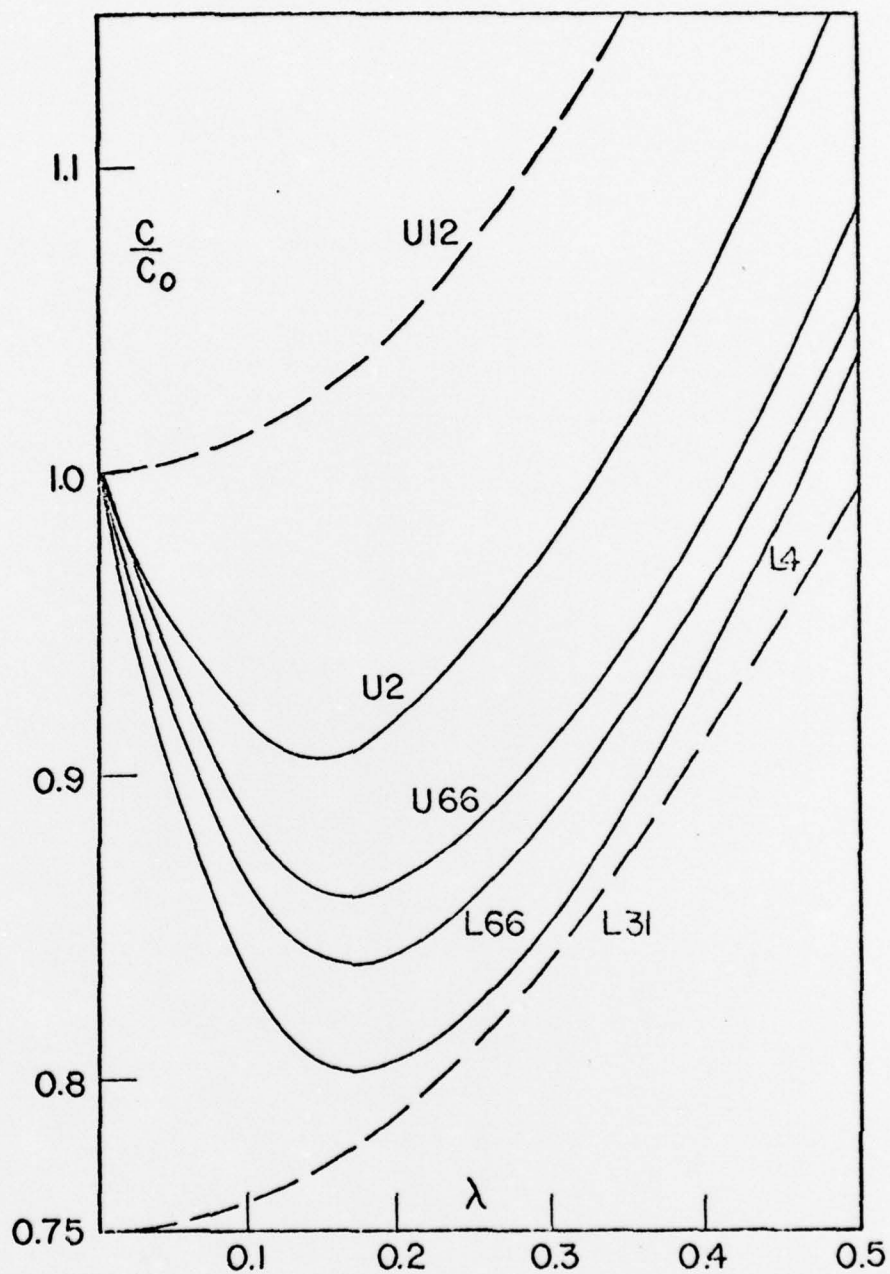| M, N	c/a = 0.3			c/a = 0.5		
	C_L/C_o	C_U/C_o	% Error[†]	C_L/C_o	C_U/C_o	% Error[†]
3, 3	1.2070	1.2805	3.05	1.6974	1.7716	2.19
4, 4	1.2260	1.2762	2.05	1.7188	1.7632	1.29
5, 3	1.2605	1.2754	0.59	1.7447	1.7674	0.65
5, 4	1.2618	1.2734	0.46	1.7457	1.7611	0.44
5, 5	1.2628	1.2732	0.32	1.7483	1.7597	0.33
6, 6	1.2642	1.2715	0.29	1.7501	1.7581	0.22

[†] % Error $= \dfrac{C_U - C_L}{2C_L} \times 100.$

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Fig. 1. Dependence of the bounds on $\lambda = \sqrt{E/G}\ c/a$ for orthotropic beams when $\nu = 1/2$ and $\mu = G/E_m = 100$.

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A numerical scheme based on polynomial approximation is presented for the calculation of upper and lower bounds for the flexibility coefficient of end-loaded orthotropic cantilever beams in a state of plane stress, through use of the principles of minimum potential energy and of minimum complementary energy. The degree-of-freedom of the two-dimensional polynomials employed along with the elastic constants and the depth-to-span ratio of the beam are external parameters of the problem.

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